

BOROUGH of MANHATTAN COMMUNITY COLLEGE

DEPARTMENT of MATHEMATICS

**MAT302-1200**

**Spring 2020**

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Project 1 - due Monday, March 2, 2020

RIEMANN SUMS, DEFINITE INTEGRAL, AREA UNDER A CURVE

**Objectives:** To calculate Riemann sums for a function. To visualize the Riemann sums as an animated series of rectangles. To explore the relationship between the area under a curve and the value of the definite integral.

**Example 1.**

Consider  $f(x) = \sin(x)$  on the interval  $[0, \frac{3\pi}{2}]$ .

a) Using Maple, evaluate  $\int_0^{\frac{3\pi}{2}} f(x) dx$ .

b) Using ApproximateInt command from the Student[Calculus1] package construct the rectangles that represents a random Riemann sum, the lower and upper sums for  $f(x)$  using  $n = 10$  subintervals. Calculate those sums.

c) Repeat (b) for  $n = 20, 40, 80, 160, 320$  for the lower and upper sums and watch those sums to converge.

d) Try  $n = 1000$  for the random, lower and upper sums (no picture needed), as well as for Trapezoidal and Simpson's sums. Compare their performances by counting the number of correct decimal digits after the decimal point.

e) Use ApproximateInt to animate the picture of convergence of a random Riemann sum to the definite integral.

**Solution of Example 1.**

```
> with(Student[Calculus1]):  
> f:=x-> sin(x);
```

$$f := x \rightarrow \sin(x)$$

(1)

a)

```
> Int(f(x), x=0..3/2*Pi);
```

$$\int_0^{\frac{3}{2}\pi} \sin(x) \, dx$$

(2)

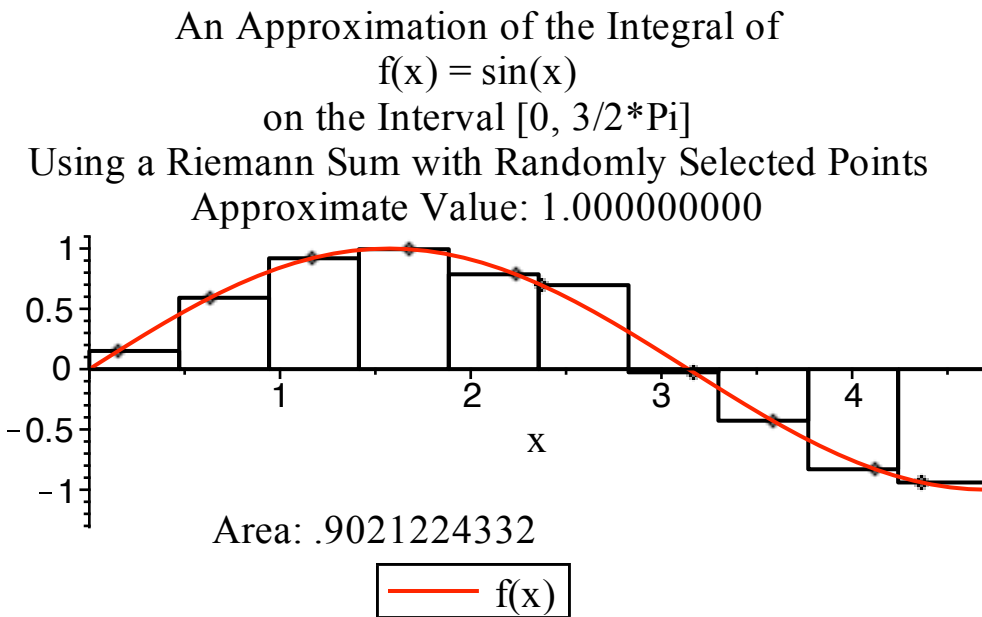
```
> evalf(%);
```

1.000000000

(3)

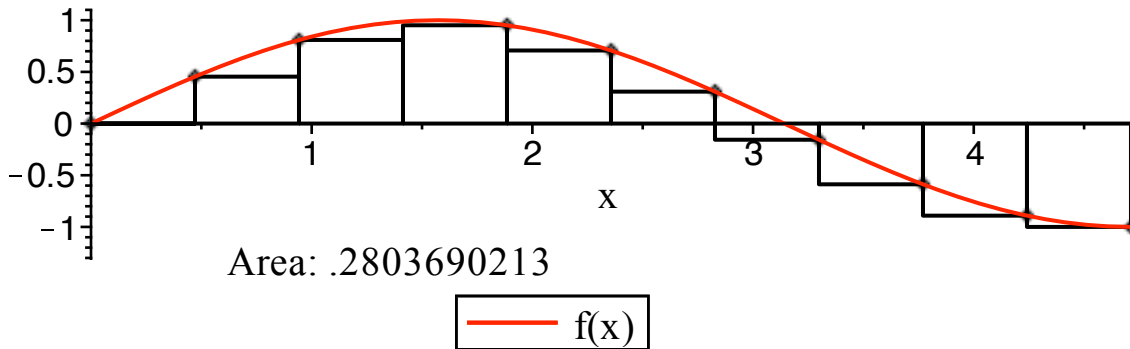
b)

```
> ApproximateInt(f(x), x=0..3/2*Pi, method=random, partition=10,
output=plot);
```



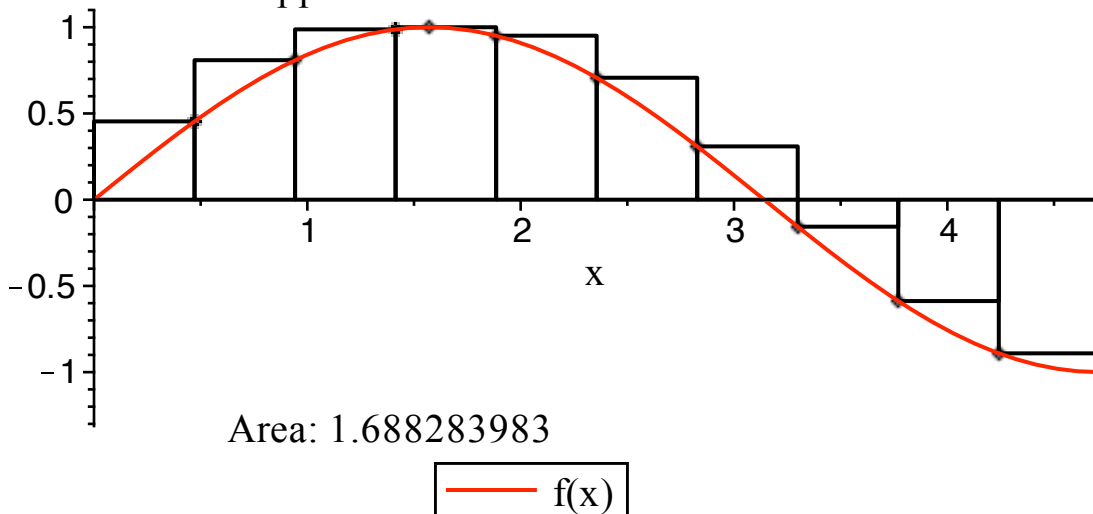
```
> ApproximateInt(f(x), x=0..3/2*Pi, method=lower, partition=10,
output=plot);
```

An Approximation of the Integral of  
 $f(x) = \sin(x)$   
on the Interval  $[0, 3/2\pi]$   
Using a Lower Riemann Sum  
Approximate Value: 1.000000000



```
> ApproximateInt(f(x), x=0..3/2*Pi, method=upper, partition=10,
output=plot);
```

An Approximation of the Integral of  
 $f(x) = \sin(x)$   
on the Interval  $[0, 3/2\pi]$   
Using an Upper Riemann Sum  
Approximate Value: 1.000000000



Note that the random approximation returns a value between the lower and upper approximations. With the number of subintervals  $n = 10$  all those approximations are still far from the "true" value 1.000000000 of the definite integral.

c) Let us repeat random, lower and upper approximations with twice as many subintervals

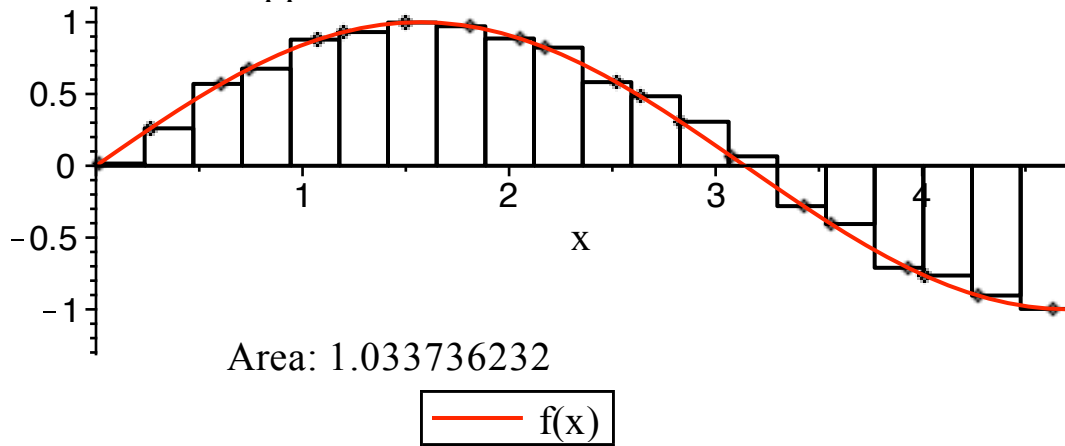
```
> ApproximateInt(f(x), x=0..3/2*Pi, method=random, partition=20,
output=plot);
```

An Approximation of the Integral of  
 $f(x) = \sin(x)$

on the Interval  $[0, 3/2 \cdot \pi]$

Using a Riemann Sum with Randomly Selected Points

Approximate Value: 1.000000000



```
> ApproximateInt(f(x), x=0..3/2*Pi, method=lower, partition=20,  
output=plot);
```

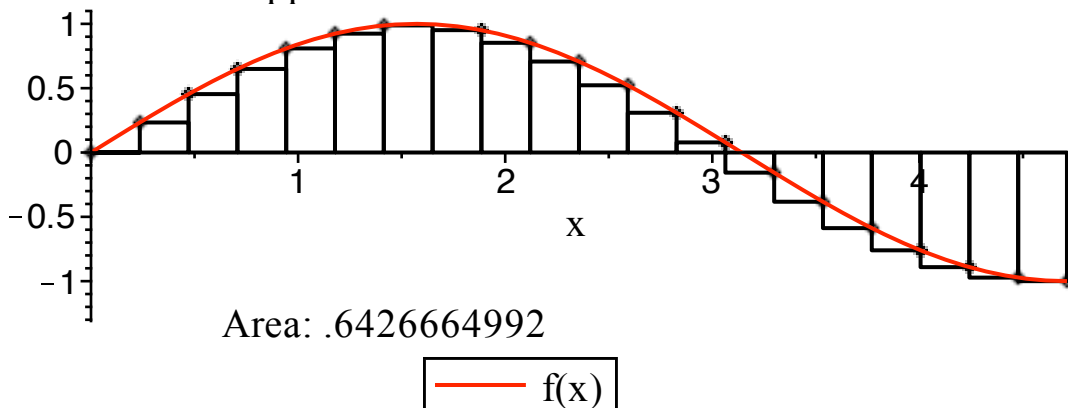
An Approximation of the Integral of

$f(x) = \sin(x)$

on the Interval  $[0, 3/2 \cdot \pi]$

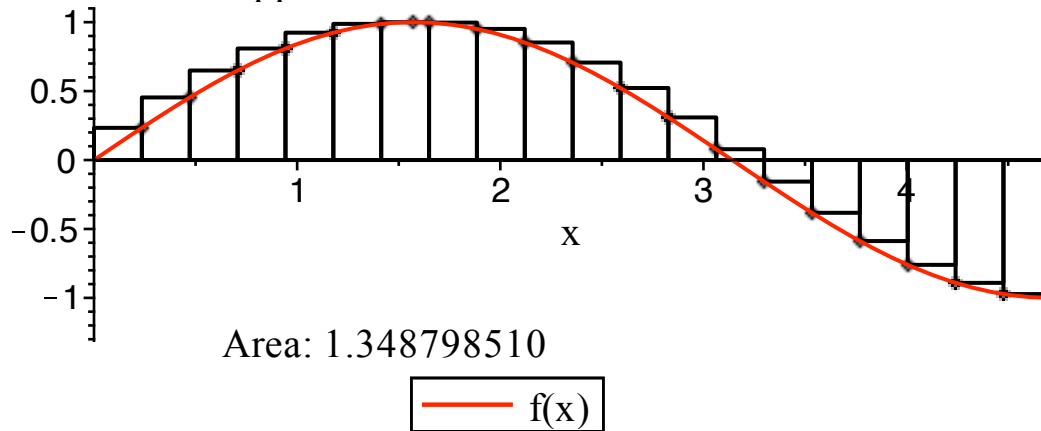
Using a Lower Riemann Sum

Approximate Value: 1.000000000



```
> ApproximateInt(f(x), x=0..3/2*Pi, method=upper, partition=20,  
output=plot);
```

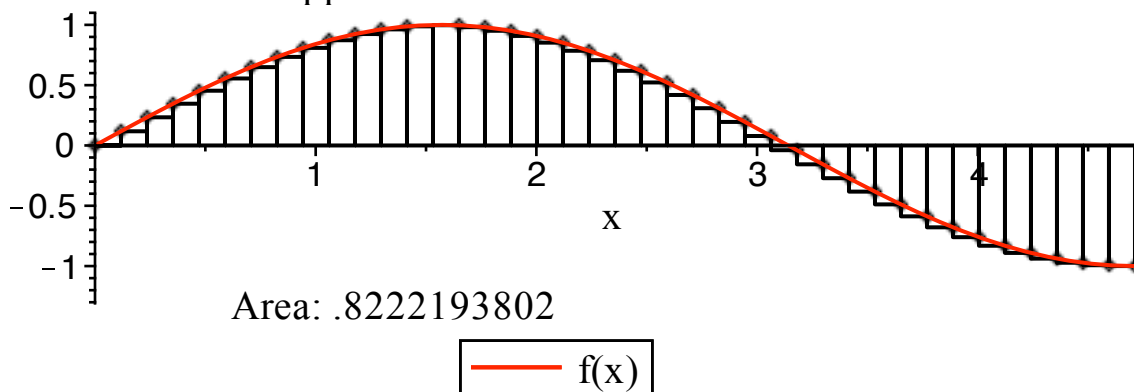
An Approximation of the Integral of  
 $f(x) = \sin(x)$   
 on the Interval  $[0, 3/2 \cdot \pi]$   
 Using an Upper Riemann Sum  
 Approximate Value: 1.000000000



Again we can see that the random approximation lies between the lower and upper ones, which are still not close to the limit. Let us double the amount of subintervals again. We concentrate on the lower and upper sums.

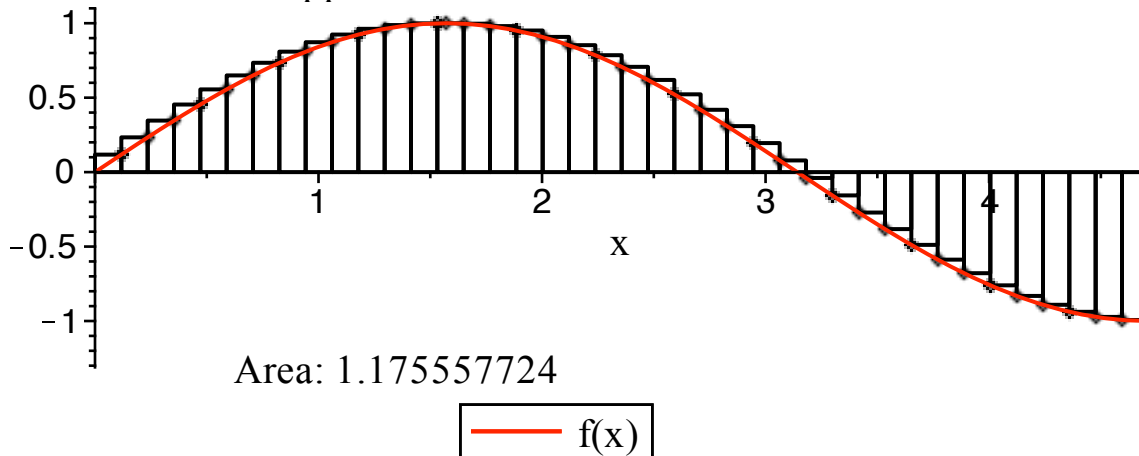
```
> ApproximateInt(f(x), x=0..3/2*Pi, method=lower, partition=40,  
output=plot);
```

An Approximation of the Integral of  
 $f(x) = \sin(x)$   
 on the Interval  $[0, 3/2 \cdot \pi]$   
 Using a Lower Riemann Sum  
 Approximate Value: 1.000000000



```
> ApproximateInt(f(x), x=0..3/2*Pi, method=upper, partition=40,  
output=plot);
```

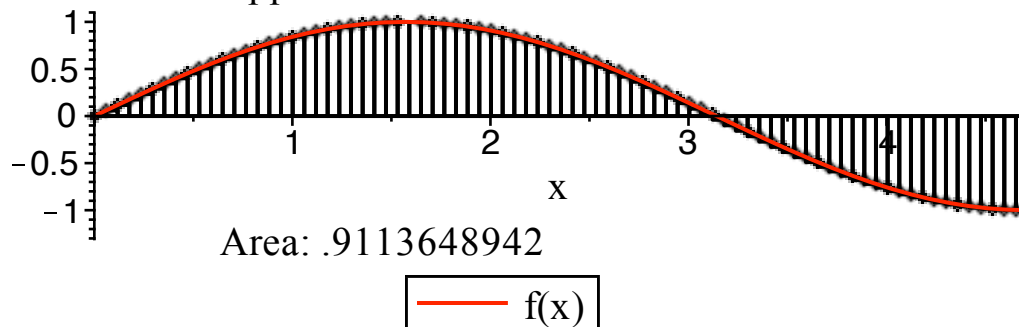
An Approximation of the Integral of  
 $f(x) = \sin(x)$   
 on the Interval  $[0, 3/2\pi]$   
 Using an Upper Riemann Sum  
 Approximate Value: 1.000000000



Still not good enough. We don't have even the first digit right. Let us try  $n = 80$ :

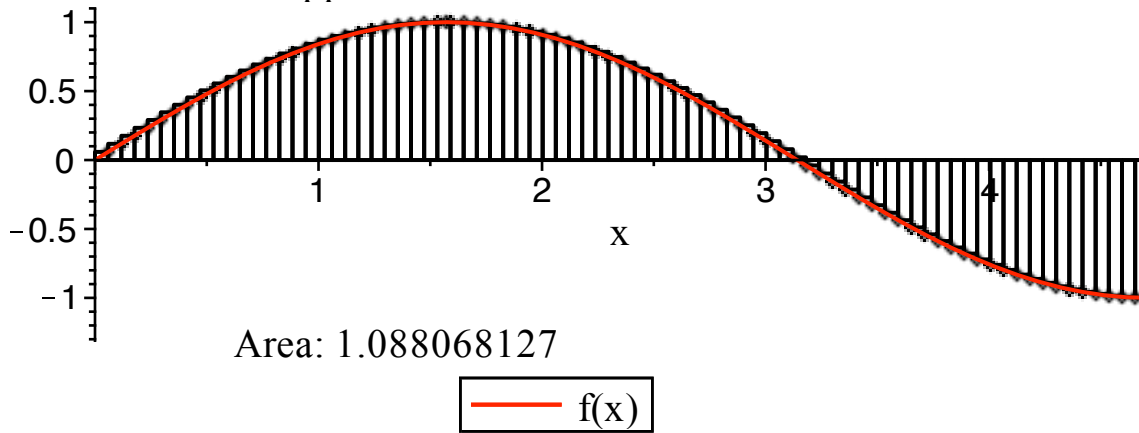
```
> ApproximateInt(f(x), x=0..3/2*Pi, method=lower, partition=80,
  output=plot);
```

An Approximation of the Integral of  
 $f(x) = \sin(x)$   
 on the Interval  $[0, 3/2\pi]$   
 Using a Lower Riemann Sum  
 Approximate Value: 1.000000000



```
> ApproximateInt(f(x), x=0..3/2*Pi, method=upper, partition=80,
  output=plot);
```

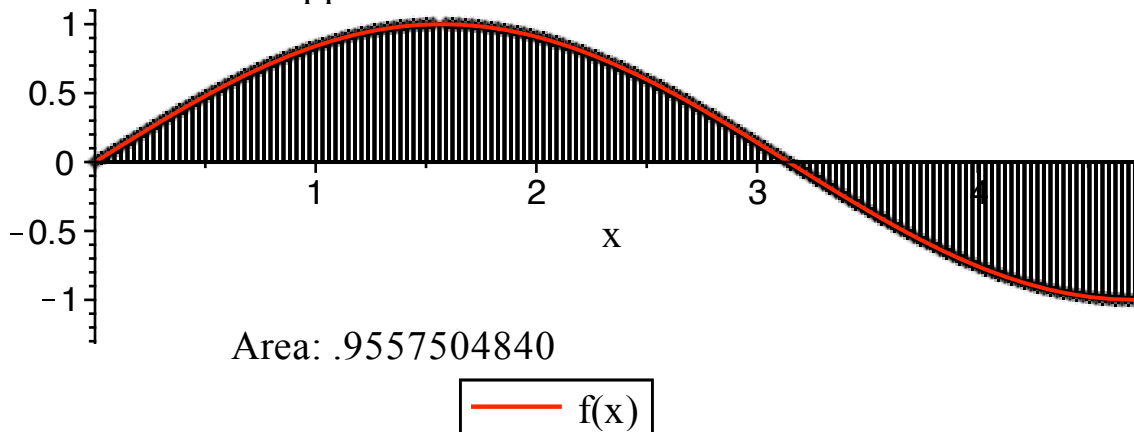
An Approximation of the Integral of  
 $f(x) = \sin(x)$   
on the Interval  $[0, 3/2\pi]$   
Using an Upper Riemann Sum  
Approximate Value: 1.000000000



$n = 160$ :

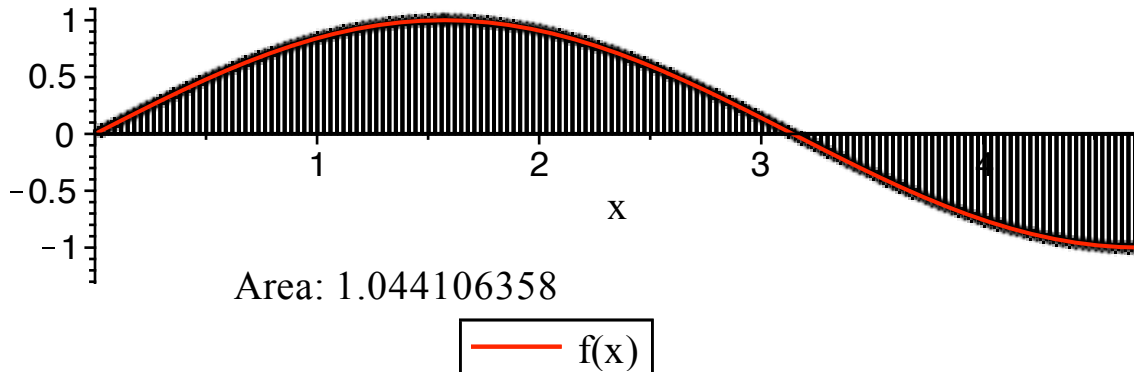
```
> ApproximateInt(f(x), x=0..3/2*Pi, method=lower, partition=160,  
output=plot);
```

An Approximation of the Integral of  
 $f(x) = \sin(x)$   
on the Interval  $[0, 3/2\pi]$   
Using a Lower Riemann Sum  
Approximate Value: 1.000000000



```
> ApproximateInt(f(x), x=0..3/2*Pi, method=upper, partition=160,  
output=plot);
```

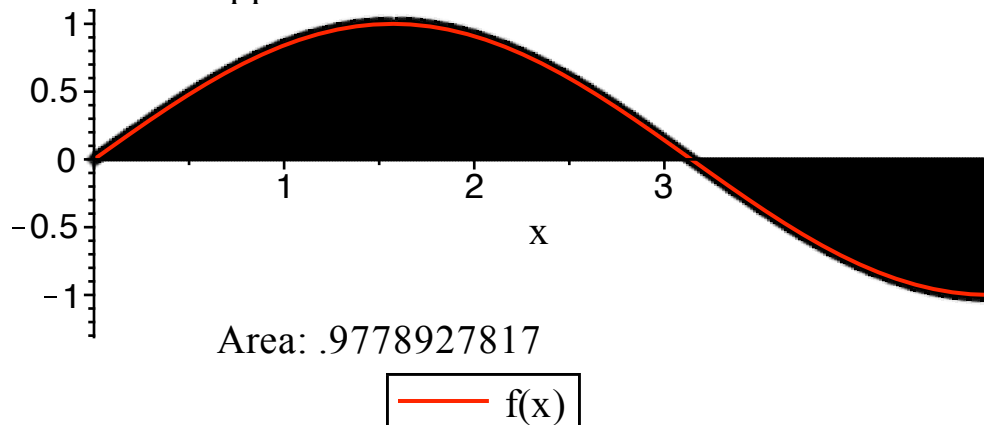
An Approximation of the Integral of  
 $f(x) = \sin(x)$   
 on the Interval  $[0, 3/2 \cdot \pi]$   
 Using an Upper Riemann Sum  
 Approximate Value: 1.000000000



We are still about 5% off the target in both approximations. Let us try  $n = 320$ :

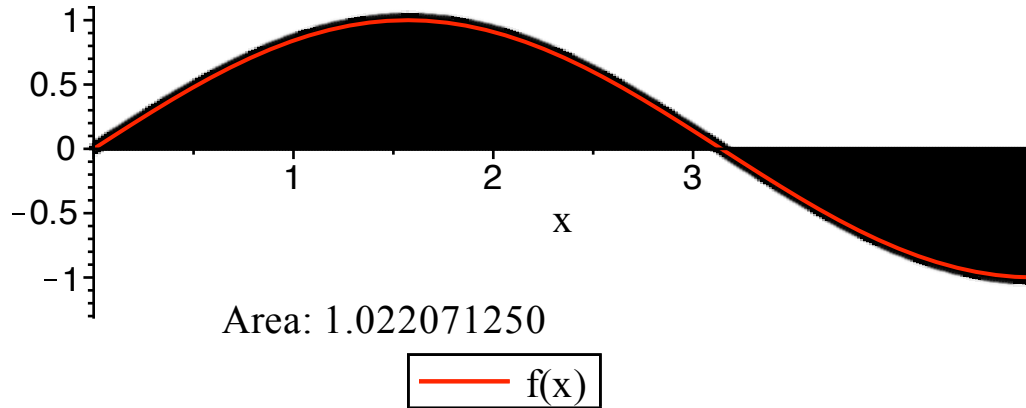
**> ApproximateInt(f(x), x=0..3/2\*Pi, method=lower, partition=320, output=plot);**

An Approximation of the Integral of  
 $f(x) = \sin(x)$   
 on the Interval  $[0, 3/2 \cdot \pi]$   
 Using a Lower Riemann Sum  
 Approximate Value: 1.000000000



**> ApproximateInt(f(x), x=0..3/2\*Pi, method=upper, partition=320, output=plot);**

An Approximation of the Integral of  
 $f(x) = \sin(x)$   
 on the Interval  $[0, 3/2 \cdot \pi]$   
 Using an Upper Riemann Sum  
 Approximate Value: 1.000000000



The partition became so dense that we can no longer distinguish individual subintervals. The approximations seem to converge to 1, but not really fast. The situation may look inconclusive.

d) Let us try  $n = 1000$ . Since a picture is no longer informative, we will switch to the output=value.

```
> ApproximateInt(f(x), x=0..3/2*Pi, method=lower, partition=1000,
  output=value);
```

0.9929295710 (4)

```
> ApproximateInt(f(x), x=0..3/2*Pi, method=upper, partition=1000,
  output=value);
```

1.007066732 (5)

Even with such a dense partition we are still getting only two decimal digits right. Now we are convinced in convergence of Riemann sums to the integral/area, but feel a need to have better approximation tools. We will try three of them: random, trapezoidal and Simpson methods, by just changing option "method" in ApproximateInt. Close the command line with ":" instead of ";" to avoid long and non-informative intermediate symbolic expressions, use evalf(%) command to compute a floating-point value.

```
> ApproximateInt(f(x), x=0..3/2*Pi, method=random, partition=1000,
  output=value):
> evalf(%)
```

1.000104153 (6)

```
> ApproximateInt(f(x), x=0..3/2*Pi, method=trapezoid, partition=1000,
  output=value):
> evalf(%)
```

0.9999981497 (7)

```
> ApproximateInt(f(x), x=0..3/2*Pi, method=simpson, partition=1000,
  output=value):
> evalf(%)
```

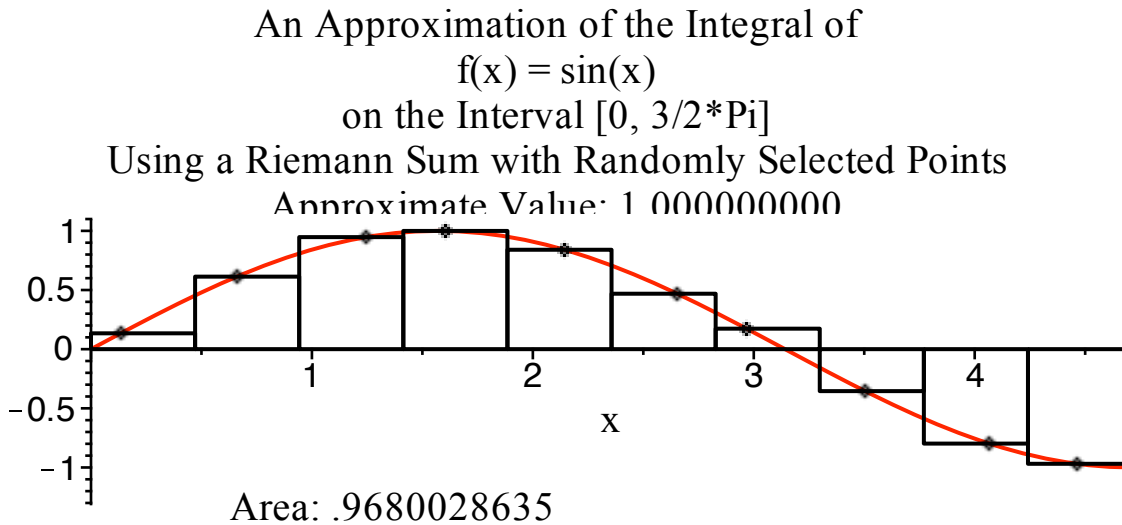
0.9999999956 (8)

We see that random method wins over both lower sums and upper sums (three correct digits vs two),

which is not a big surprise though. Lower and upper sums realize the worst cases of deviations from the limit, whereas in the random choice of points errors average up. The trapezoidal method won over random (five correct digits), and Simpson method was even better and gave eight right digits under the same partition.

Finally, we will run ApproximateInt in the animation mode to visualize the approximation of an area by Riemann sums. Run the animation in a continuous and slow mode to see more details.

```
> ApproximateInt(f(x), x=0..3/2*Pi, method=random, partition=10,
  output=animation, iterations=7);
```



**Example 2.** Given  $f(x) = x^3 - 4x$ ,

- find the definite integral of  $f(x)$  from  $x = -2$  to  $x = 2$ ;
- plot the function  $f(x)$ ;
- find the area between the curve  $f(x)$  and the  $x$ -axis from  $x = -2$  to  $x = 2$ ;
- determine whether the values in a) and c) are equal, and explain why.

Solution of Example 2.

```
> f:=x->x^3 - 4*x;
```

$f := x \rightarrow x^3 - 4x$  (9)

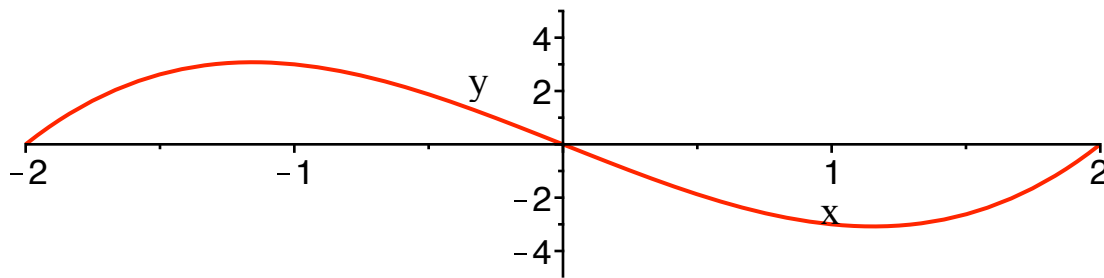
a)

```
> definiteintegral:=int(f(x),x = -2..2);
```

$\text{definiteintegral} := 0$  (10)

b)

```
> plot(f(x), x = -2..2, y = -5..5);
```



c)

```
> area:=int(f(x),x = -2..0) + abs(int(f(x),x = 0..2));
```

(11)

```
area := 8
```

d) The definite integral does not equal the area under the curve. Since the part of the curve below the  $x$ -axis is equal to the part of the curve above the  $x$ -axis, the definite integral will be zero.

Example 3.

Consider  $f(x) = \sin(x^2)$ .

(a) Find  $g(x)$  as the definite integral of function  $f$  from 0 to  $x$  for  $x$  ranging from 0 to  $2\pi$ . Verify that  $g(x)' = f(x)$ .

(b) Find  $h(x)$  as the definite integral of function  $f$  from 1 to  $x$  for  $x$  ranging from 0 to  $2\pi$ . Verify that  $h(x)' = f(x)$ .

(c) Plot  $g(x)$  and  $h(x)$  in the same window.

(d) Prove analytically that the difference  $g(x) - h(x)$  is a constant on  $[0, 2\pi]$ . Find this constant with the accuracy of seven decimal digits.

Solution of Example 3.

```
> with(plots):
```

The integral of  $f(x) = \sin(x^2)$  cannot be found by any formulas. Because  $\sin(x^2)$  is continuous, the Second Fundamental Theorem of Calculus yields that an antiderivative of  $f(x)$  on  $[0, 2\pi]$  can be given by  $g(x) = \int_0^x \sin(t^2) dt$ . Let us verify that  $g(x)' = f(x)$ .

```
> f := x -> sin(x^2);
```

(12)

```
f := x -> sin(x^2)
```

```
> g := x -> int(sin(t^2), t = 0..x);
```

(13)

```
g := x -> int(sin(t^2), t = 0..x)
```

```
> diff(g(x),x);
```

(14)

```
sin(x^2)
```

Likewise, we define  $h(x) = \int_1^x \sin(t^2) dt$  and verify that  $h(x)' = f(x)$ .

```
> h := x -> int(sin(t^2), t = 1..x);
```

(15)

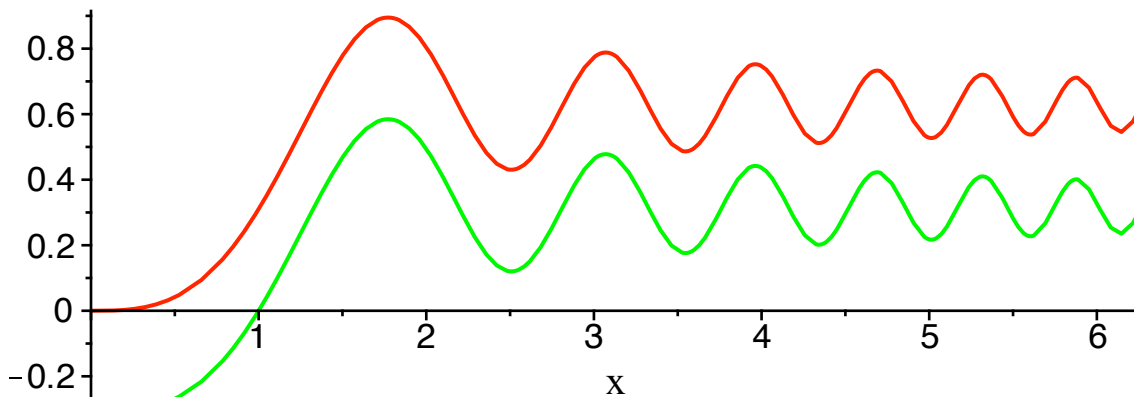
```
h := x -> int(sin(t^2), t = 1..x)
```

```
> diff(h(x),x);
```

$\sin(x^2)$

(16)

```
> plot([g(x),h(x)],x=0..2*Pi);
```



Conjecture: these functions differ by a constant approximately equal to 0.3. To verify this Conjecture we notice that the difference  $g(x) - h(x)$  is equal to

$$\int_0^x \sin(t^2) dt - \left( \int_1^x \sin(t^2) dt \right) = \int_0^x \sin(t^2) dt + \int_x^1 \sin(t^2) dt = \int_0^1 \sin(t^2) dt$$

The latter is a definite integral which does not depend on any variable and thus is a constant. It remains now to evaluate this constant using Maple definite integration package.

```
> Int(sin(t^2), t = 0..1);
```

$$\int_0^1 \sin(t^2) dt$$

(17)

```
> evalf(%);
```

0.3102683017

(18)

With the accuracy of seven decimal digits this constant is equal to 0.3102683.

## ASSIGNMENT

### Problem 1.

Consider function  $f(x) = x \sin(3x)$  on the interval  $[0, \frac{\pi}{2}]$ . Analyze approximations to the definite integral of this function on the given interval by following the sequences of the steps from (a) through (e) of Example 1 above (do not forget to make appropriate changes in the data!)

### Problem 2.

Given  $f(x) = 2x^3 - x^2$ ,

- plot the function;
- find the area between the curve and the  $x$ -axis from  $x = -1$  to  $x = 2$ ;
- find the definite integral of the function from  $x = -1$  to  $x = 2$ ;
- determine whether the values in b) and c) are equal; explain the result.

### Problem 3.

Consider  $f(x) = \cos(x^2)$ .

- a) Find  $g(x)$  as the definite integral of function  $f$  from  $-\pi$  to  $x$  for  $x$  ranging from  $-\pi$  to  $0$ . Verify that  $g(x)' = f(x)$  on the interval  $[-\pi, 0]$ .
- b) Find  $h(x)$  as the definite integral of function  $f$  from  $-1$  to  $x$  for  $x$  ranging from  $-\pi$  to  $0$ . Verify that  $h(x)' = f(x)$ .
- c) Plot  $g(x)$  and  $h(x)$  on the same graph.
- d) Prove analytically that the difference  $g(x) - h(x)$  is a constant on  $[-\pi, 0]$ . Find this constant with the accuracy of five decimal digits.